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SECOND-ORDER ERRORS IN LINEARIZED VELOCITY CORRECTIONS AND THEIR EFFECTS ON THE RATE OF CONVERGENCE OF ITERATIVE SOLUTIONS

by John D. McLean

Ames Research Center
Moffett Field, Calif.

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SUMMARY

The use of a fast integration scheme, such as Danby's method, allows a choice of methods for the on-board computation of linearized velocity corrections that is impractical with slower integration methods. This report deals with the problem of errors, caused by nonlinearity, in these velocity corrections and the effect of those errors on the choice of the computation method.

Two methods of computing linearized corrections are considered. The first, referred to as the "linear" method, computes the velocity correction as a linear function of position and velocity deviations at the time of the correction. The second method, referred to as the "half-linear" method, computes the trajectory ahead to the terminal time and then determines velocity correction as a linear function of the actual terminal error.

This report has four parts: (1) The linear and half-linear guidance laws are presented together with formulas for predicting the errors in them caused by nonlinearity. These formulas are based on the assumption that the errors are adequately represented by second-order terms. (2) Second-order errors are presented for both guidance methods on conic trajectories with eccentricities of 0.2, 0.98, and 2.0, and the errors for the two guidance methods are compared. (3) A simulated lunar mission is used as an example to show how the data from the conic trajectories can be used to give a qualitative idea of the relative performance of the linear and half-linear methods on an n-body trajectory. (4) The relative advantage of using the linear or half-linear method to compute the first velocity correction when iteration is used to reduce the residual error is investigated. Also, the effect of neglecting perturbing forces in calculating the state transition matrix is considered.

The two-body data show that the linear method of velocity correction is usually superior, the second-order errors often being several orders of magnitude less than those for the half-linear method. If a single conic is not sufficient to approximate the actual trajectory, the errors still follow the same trends as the two-body data and the linear method is usually superior. The use of the linear method to start an iteration method usually results in more rapid convergence than when the half-linear method is used.

INTRODUCTION

This study is part of an effort at Ames Research Center to simplify the computations required for midcourse guidance and navigation by on-board systems. The purpose of such simplification is to allow reduction of the size and power requirements of the on-board computer with a consequent increase in its reliability. A reduction in the computation time is also desirable because the time saved can often be traded for computer storage, hence, size.

In reference 1 it is shown that Danby's method of integrating the orbital equations of motion is particularly suited for use in on-board navigation systems. For a given accuracy, this integration method requires much less time and storage than Cowell's method and is probably superior to Encke's method. The superiority of Danby's method results, in part, from the fact that the transition matrices used in trajectory estimation and linearized guidance are a by-product of the integration process. The trajectory is approximated over small-time intervals by conics, and the two-body transition matrix over each approximating conic is used in integrating the accelerations that perturb the spacecraft from it. For guidance purposes, the resulting sequence of matrices may be multiplied together to give a good approximation to the matrix that would be obtained by integrating the two-body variational equations along the n-body trajectory.

If the integration method is fast, then either implicit or explicit guidance may be used. (See reference 2 for a detailed discussion of implicit and explicit guidance.) If the integration is slow, explicit guidance is impractical, but implicit guidance may be used as in reference 3.

This study is restricted to linearized guidance, and we will refer to linearized explicit guidance as the "half-linear" method and linearized implicit guidance as the "linear" method. With the half-linear method, the best estimate of the actual trajectory is integrated ahead to the final time, and the velocity correction is computed as a linear function of the terminal error. In the linear method, the velocity correction is computed as a linear function of the position and velocity deviations from a reference trajectory at the time of the correction.

The main purpose of this report is to compare the errors that arise when these two different methods are used for computing linearized velocity corrections. Danby's integration method was used in this study, but the results are applicable when other integration schemes are used. Two potential error sources are considered: the first, with which most of the report is concerned, is nonlinearity, and it is present in any linearized guidance scheme. The second is the inaccuracy that arises from neglecting the perturbing forces in the computation of the transition matrices.

The first portion of this report presents the linear and half-linear guidance laws and equations for predicting the errors due to the nonlinearity under the assumption that the second-order terms are a reasonable measure of those errors. Then, predicted second-order errors are presented for three

conics with eccentricities representative of trajectories likely to be encountered in space missions. These data show the effects of the magnitude of initial condition errors, true anomaly at the time of the correction, and the arc length between the correction and terminal points.

Next the results for some guidance problems simulated using a translunar and a transearth trajectory are compared with the two-body results. Additional data from these guidance problems show how the choice of the linear or half-linear method for the first velocity correction affects the rate of convergence when the residual errors are reduced by iteration. The errors due to neglecting the perturbing forces in calculating the transition matrices are also considered.

All the data presented were obtained using fixed-time-of-arrival guidance and initial condition errors having Gaussian distribution and zero mean.

NOTATION

A	matrix of first partial derivatives of $\overline{P}$ with respect to initial position
B	matrix of first partial derivatives of $\overline{P}$ with respect to initial velocity
B^+	pseudoinverse of B
G	matrix of second partial derivatives of position and velocity with respect to their initial values
H	matrix used in calculating statistical information
$H_1,H_2,$ H_3,H_4	submatrices of H
I	identify matrix
$\overline{P}$	vector of parameters to be controlled
$\overline{p}$	vector of first-order deviations in $\overline{P}$
$\overline{r}$	position deviation vector
r	magnitude of $\overline{r}$
rms	root mean square
Tr	trace of a matrix

$\bar{v}$	velocity deviation vector
v	magnitude of $\bar{v}$
X,Y,Z	Cartesian components of position vector
α	arbitrary component of $\bar{p}$
δ	second-order deviation
θ	true anomaly
$\Delta\theta$	change in true anomaly from correction to terminal point
Φ	state transition matrix for linearization around actual trajectory
ϕ	state transition matrix for linearization around reference trajectory
$\Phi_1, \Phi_2,$ ϕ_1, ϕ_2	submatrices of Φ and ϕ, respectively

Subscripts

D	desired
f	final
H	half-linear
L	linear
o	initial
$\lVert\cdot\rVert_E$	Euclidean norm of a matrix (root sum square of all its elements)

SECOND-ORDER ERRORS FOR LINEARIZED GUIDANCE

This section of the report presents the linear and half-linear guidance laws, the equations used for predicting the second-order errors in these guidance methods, and the assumptions made in deriving the equations. The derivations of the equations for the second-order errors are given in appendix A.

Linear guidance depends on expressing one trajectory in terms of a Taylor series expansion about another and neglecting all but the first-order terms.

It is assumed in this analysis that:

1. The second-order terms give a good measure of the total residual error due to nonlinearity.

2. The state of the spacecraft (position and velocity vectors) at the time of the correction is known exactly.

3. The components of the position and velocity deviations from a precomputed reference trajectory at the time of the correction are independent and Gaussian with zero mean. Also, the standard deviation in each component of position and velocity, respectively, is the same.

4. Only fixed-time-of-arrival guidance will be used.

Two different methods of computing the linear corrections are considered. The first method uses the definition of linear impulsive guidance given in reference 4; that is, the velocity correction is computed as a linear function of the deviations of the present state (at the time of the velocity correction) from a given reference state. This method of computation will be referred to as the "linear" method. The second method of computation consists in defining the velocity correction as a linear function of the deviations of the parameters to be controlled from their desired values at the terminal time. Since the complete nonlinear relationship between the initial state deviations and the controlled parameters are used to determine the latter, this method will be referred to as the "half-linear" method.

The difference between the two methods of guidance is illustrated in figure 1. At the time of the velocity correction, the spacecraft position and velocity vectors differ from the reference values by $\bar{r}_o$ and $\bar{v}_o$, respectively, and if no correction is made, the spacecraft will miss the desired terminal point by the error vector $\bar{r}_f$. Therefore, we wish to compute a change in velocity such that the corrected trajectory, indicated by the dashed line, will terminate at the desired point.

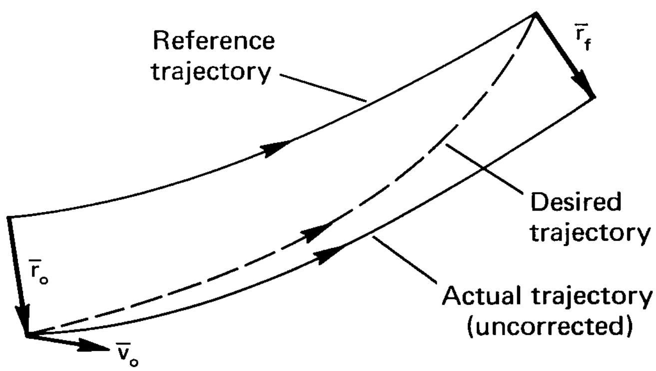


Figure 1.- Fixed-time-of-arrival guidance methods.

The linear method of computation requires a knowledge of $\bar{r}_o$, $\bar{v}_o$, and the transition matrix, ϕ, which gives $\bar{r}_f$ as a linear function of those two quantities. (See appendix B.) The terminal error vector can then be written as

$$\bar{r}_f = \phi_1 \bar{r}_o + \phi_2 \bar{v}_o + \delta \bar{r}_f \tag{1}$$

where ϕ_1 and ϕ_2 are submatrices of ϕ, and $\delta \bar{r}_f$ represents the nonlinear terms in the Taylor series expansion of the actual trajectory about the reference trajectory.

To compute the velocity correction for the linear method, we assume that $\delta \bar{r}_f$ is zero, and solve equation (1) for the velocity correction $\bar{v}_L$, which, when added to $\bar{v}_o$, will cause $\bar{r}_f$ to be zero. The result is

$$\bar{v}_L = -\phi_2^{-1} \phi_1 \bar{r}_o - \bar{v}_o \tag{2}$$

If this correction is added to the actual velocity and the corrected trajectory is expanded in a Taylor series about the reference, the first-order error is zero and the residual error will be denoted by $\delta \bar{r}_L$.

For the half-linear method, the actual trajectory is integrated ahead to the final time, and the actual terminal position vector is subtracted from the desired one to give $-\bar{r}_f$. This deviation can be expanded in a Taylor series about the actual trajectory to give

$$-\bar{r}_f = \Phi_2 \bar{v}_D + \delta \bar{r}_D \tag{3}$$

Here Φ_2 is a submatrix of the transition matrix Φ obtained by linearizing about the actual trajectory, $\bar{v}_D$ is the exact velocity correction required, and $\delta \bar{r}_D$ represents the nonlinear part of the expansion. If $\delta \bar{r}_D$ is assumed to be zero, that is, all of $-\bar{r}_f$ is produced by the linear term alone, then the velocity correction for the half-linear method is

$$\bar{v}_H = -\Phi_2^{-1} \bar{r}_f \tag{4}$$

If this correction is added to the actual velocity and the corrected trajectory is expanded about the actual trajectory, the terminal deviation is

$$\bar{r}_H = -\bar{r}_f + \delta \bar{r}_H \tag{5}$$

where $\delta \bar{r}_H$ is the residual error in the expansion of the corrected trajectory about the actual.

If it is assumed that terms of higher than second order may be neglected, then each component of $\delta \bar{r}_L$ is given by a quadratic form in $\bar{r}_o$ and each component of $\delta \bar{r}_H$ by a quadratic form in $\bar{v}_H$. The mean square error (derived in appendix A) in the X component of the second-order error is

$$\left.\begin{aligned} E\left(\delta X_L^2\right) &= \frac{1}{2}\,\sigma_r^4\|H_L\|_E^2 + \frac{1}{4}\,\sigma_r^4\,(\mathrm{Tr}H_L)^2 \\ E\left(\delta X_H^2\right) &= \frac{1}{2}\left[\sigma_r^4\|H_1\|_E^2 + 2\sigma_r^2\sigma_v^2\|H_2\|_E^2 + \sigma_v^4\|H_4\|_E^2\right] + \frac{1}{4}\left[\mathrm{Tr}\left(\sigma_r^2 H_1 + \sigma_v^2 H_4\right)\right] \end{aligned}\right\} \quad (6)$$

where $\|H\|_E$ denotes the Euclidian norm of the matrix H. The parameters σ_r and σ_v are the standard deviations in the components of $\overline{r}_o$ and $\overline{v}_o$, respectively, and the matrices H_L, H_1, H_2, and H_4 are defined in appendix A.

The rms magnitude (δr_L or δr_H) of the second-order error vector is given by

$$\left.\begin{aligned} (\delta r_L)_{rms} &= \left[E\left(\delta X_L^2\right) + E\left(\delta Y_L^2\right) + E\left(\delta Z_L^2\right)\right]^{1/2} \\ (\delta r_H)_{rms} &= \left[E\left(\delta X_H^2\right) + E\left(\delta Y_H^2\right) + E\left(\delta Z_H^2\right)\right]^{1/2} \end{aligned}\right\} \quad (7)$$

TWO-BODY RESULTS

The rms second-order errors were computed (see appendix B) for three different conic trajectories with eccentricities of 0.2, 0.98, and 2.0. The initial conditions for the high eccentricity ellipse are those for a circumlunar trajectory, while the hyperbola uses the same initial conditions except that the magnitude of the velocity has been increased to give the desired eccentricity. For the low eccentricity ellipse, the semimajor axis, and hence the period, is kept the same as that of the high eccentricity ellipse.

The eccentricities of 0.98 and 2.0 are typical of the geocentric and selenocentric portions of lunar trajectories such as those used for the sample guidance problems in the next section. The low (0.2) eccentricity ellipse is included to show how the results vary with eccentricity, and is roughly comparable to the heliocentric portion of a Mars or Venus trajectory.

Discussion of Data

It can be seen from equations (6) and (7) that the rms second-order error for the linear method is a function only of the initial deviation and increases linearly with σ_r^2. Therefore, the errors are given only for $\sigma_r = 1$ km, and those for other values of σ_r can be obtained by multiplying the errors for $\sigma_r = 1$ km by the square of the new value of σ_r. The half-linear error is a function of both σ_r^2 and σ_v^2, and data for this method were obtained by using $\sigma_r = 1$ km with $\sigma_v = 0$, 0.1, and 1.0 m/sec. These data can be scaled in the same way for other values of σ_r and σ_v provided

σ_V/σ_r remains constant. For example, if the rms errors for $\sigma_r = 1$ km with $\sigma_V = 0$, 0.1, and 1.0 m/sec are multiplied by 100, the results are the errors for $\sigma_r = 10$ km with values of σ_V of 0, 1, and 10 m/sec, respectively.

For the high (0.98) eccentricity ellipse, six values of initial true anomaly, θ_o, were used (0°, 104°, 158°, 180°, 203°, and 256°) and they are indicated on the scale drawing of the ellipse in figure 2. A translunar or transearth trajectory can be approximated fairly well outside the sphere of influence of the Moon by such an ellipse, and the approximate true anomalies of entrance into or exit from the sphere of influence are indicated. These true anomalies have no particular significance in this section of the report and are included only to help relate the two-body results to the simulated lunar mission considered later.

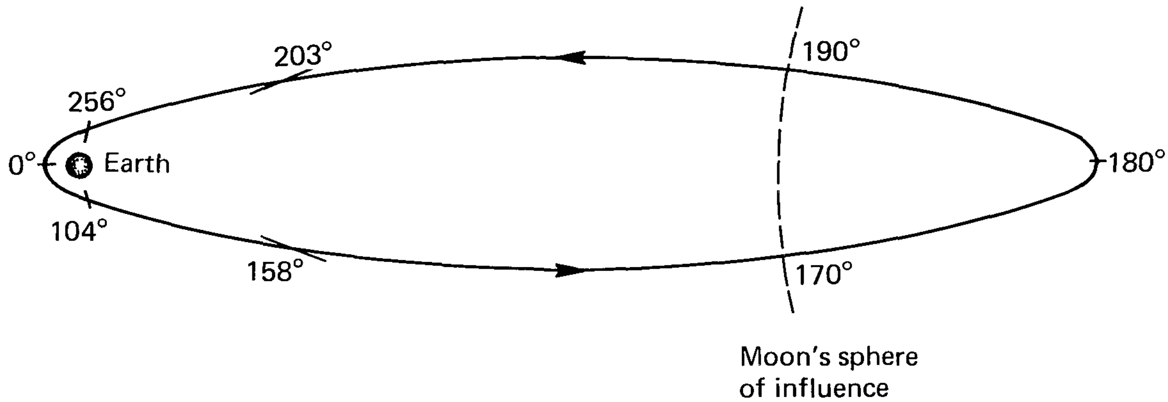


Figure 2.- True anomalies of correction points and spheres of influence on high eccentricity ellipse.

The rms second-order error for the high (0.98) eccentricity ellipse is plotted as a function of the true anomaly, θ_f, at the terminal time in figures 3(a) through 3(f). The data for the correction point at perigee are presented in figure 3(a). Initial velocity errors for this case have little influence, and the curves for $\sigma_V \neq 0$ are omitted. The rms second-order error does not exceed 1 km until the true anomaly reaches 165°, and at 170° (approximately the Moon's sphere of influence on the corresponding translunar trajectory) it is about 10 km.

For the incremental true anomaly, $\Delta\theta$, less than 180°, the error in the half-linear method is slightly less than for the linear method. (For $\sigma_V = 1$, the two are almost equal.) Since ϕ_2 is singular at $\Delta\theta = 180°$, the velocity correction and second-order error become infinite. As $\Delta\theta$ increases beyond 180°, the half-linear error rapidly becomes several orders of magnitude larger than that for the linear method.

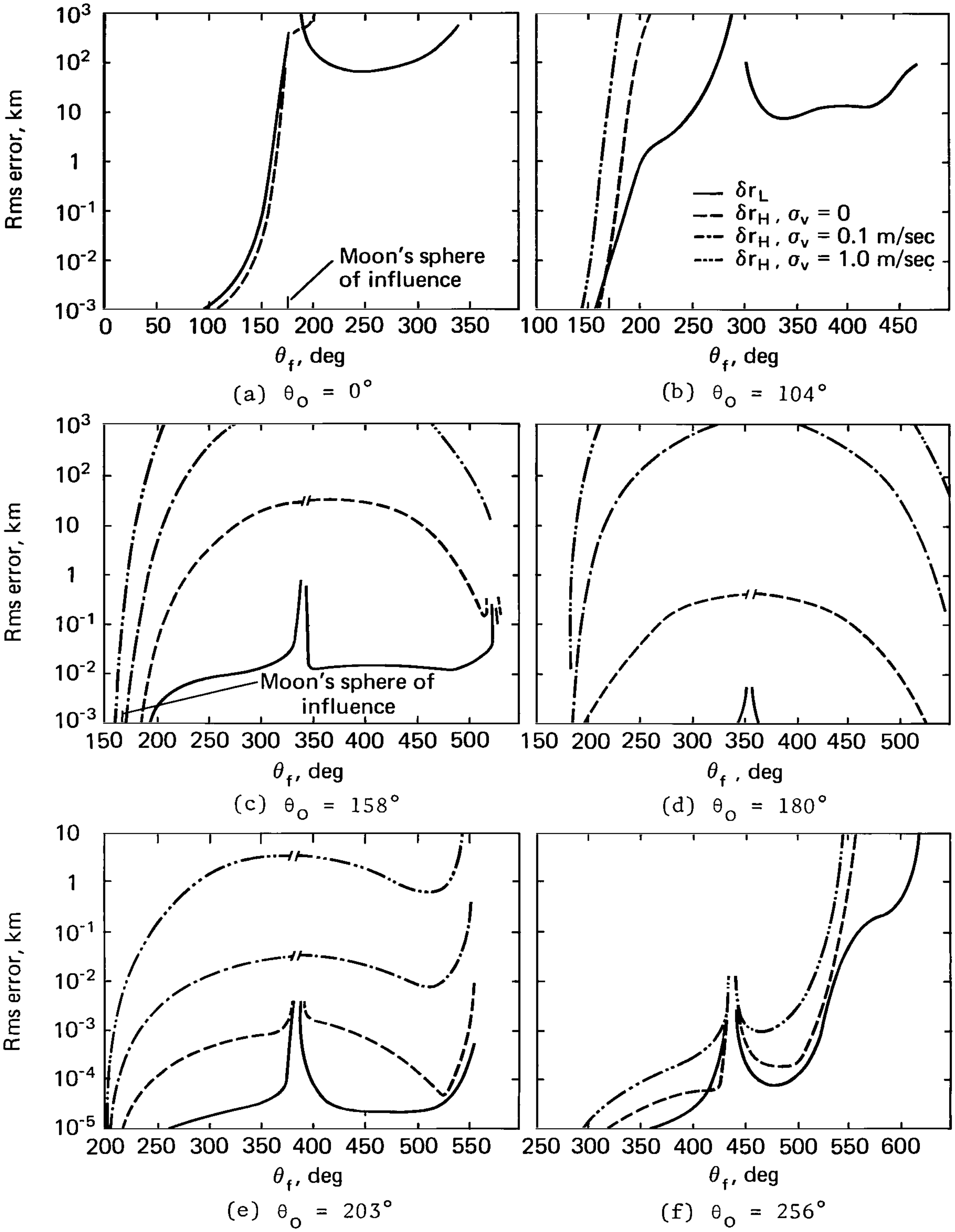


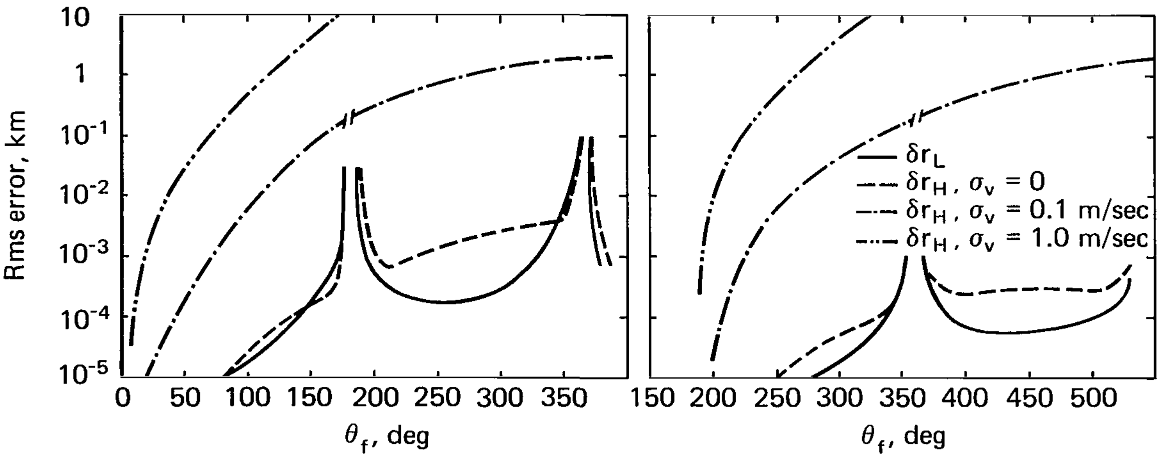
Figure 3.- Rms second-order errors for high eccentricity ellipse with σ_r = 1 km.

Figure 3(b) shows the data for a velocity correction made at a true anomaly of 104°. This point occurs about 0.6 hour after perigee, about the earliest likely point for the first midcourse velocity correction on a translunar trajectory. Initial velocity errors are quite important in the half-linear method for this value of θ_o, although the curve for $\sigma_V = 0.1$ m/sec is almost the same as for $\sigma_V = 0$ and has been omitted. The half-linear errors at the Moon's sphere of influence are 0.06 km for $\sigma_V = 0$ and 1.5 km for $\sigma_V = 1.0$ m/sec, and they exceed the error for the linear method by factors of 3 and 75, respectively. For larger values of θ_f these factors reach several orders of magnitude. For small values of $\Delta\theta$, the error for the linear method exceeds the half-linear error for $\sigma_V = 0$, and eventually exceeds that for $\sigma_V = 1.0$ m/sec when the errors for both methods are below the scale of the graph. However, the errors are very small in this region and the error for the linear method is never greater than three times the half-linear error. In this figure, the errors for a given value of θ_f are much less than in figure 3(a) where the correction is made at perigee. Note, however, that the same values of σ_r and σ_V were used in computing these curves as for those in figure 3(a). Actually, the magnitudes of the deviations from the reference trajectory will increase if the correction is delayed until $\theta_o = 104°$, and the resulting second-order errors will also be larger.

Figure 3(c) presents the data for $\theta_o = 158°$. This true anomaly occurs about 13 hours after perigee and represents the latest time likely for the first midcourse correction. At the Moon's sphere of influence, only the half-linear error for $\sigma_V = 1.0$ m/sec is above the scale of the graph, but data for the other cases were obtained. The half-linear error at this value of θ_f exceeds that for the linear method by an order of magnitude for $\sigma_V = 0$ by three and five orders of magnitude, respectively, for $\sigma_V = 0.1$ and 1.0 m/sec. The trend at low values of $\Delta\theta$ for the half-linear errors to be smaller than the linear errors was also noted, but only where the errors for both methods are extremely small.

Note that the initial velocity errors now have a very large effect on the half-linear errors and are the dominant source of error even for $\sigma_V = 0.1$ m/sec. Also, as θ_o increases, the peak in the curves near $\Delta\theta = 180°$ becomes very narrow.

Figures 3(d) through 3(f) show similar data for the return portion of the high (0.98) eccentricity ellipse. The Moon's sphere of influence on a return trajectory from the Moon corresponds to a true anomaly of about 190°. For this portion of the ellipse, the error for the half-linear method is again larger except for very small values of $\Delta\theta$ and near the singular points. When the correction is made at large distances from perigee, the half-linear error exceeds that for the linear method by several orders of magnitude, but this factor decreases as the correction point approaches perigee. Also, as the correction point approaches perigee the initial velocity deviations become relatively less important. In figure 3(f) the curve for $\sigma_V = 0.1$ m/sec is so close to that for $\sigma_V = 0$ that it has been omitted.



(a) $\theta_o = 0°$ (b) $\theta_o = 180°$

Figure 4.- Rms second-order errors for low eccentricity ellipse with $\sigma_r = 1$ km.

Figures 4(a) and 4(b) show the data for the low (0.2) eccentricity ellipse. Since this orbit is so nearly circular, the transition matrices, second partial derivatives, and the resulting second-order errors for a given change in true anomaly are nearly independent of θ_o. Therefore, only the curves for $\theta_o = 0°$ and 180° are presented; the data for other values of θ_o are between them. These curves have the same general characteristics as those for the high (0.98) eccentricity ellipse; that is, the linear method is the superior one over the greater part of the trajectory; exceptions occur near the singular points and for very small values of $\Delta\theta$. The half-linear errors may exceed those for the linear method by a factor of several orders of magnitude, but this factor does not reach the extreme values encountered in the high (0.98) eccentricity case.

We can conclude from the data for the two ellipses that the linear method will usually be superior for elliptic trajectories, the errors being several orders of magnitude smaller than for the half-linear method. Exceptions occur near the singular points, where corrections are unlikely, and for the lower values of $\Delta\theta$ where the errors are so small as to be insignificant. This range of $\Delta\theta$ is usually very small, but it increases as θ_o approaches perigee or as eccentricity increases and reaches 180° for the high (0.98) eccentricity ellipse with $\theta_o = 0$. However, the errors for the linear method never exceed those for the half-linear method by large amounts, and it seems reasonable to use the linear method as the sole method of correction.

The hyperbola is drawn to scale in figure 5, and the true anomalies of the correction points are indicated. The radial distance from the Earth at $-115°$ is 1.2×10^5 km, at $-119.8°$ the distance is 2.6×10^6 km, which could not be shown in the figure. The Earth's sphere of influence relative to the Sun is at about $\pm115°$. If the Moon were the central body, the true anomaly of its sphere of influence relative to the Earth would be at about 111° for the same semimajor axis.

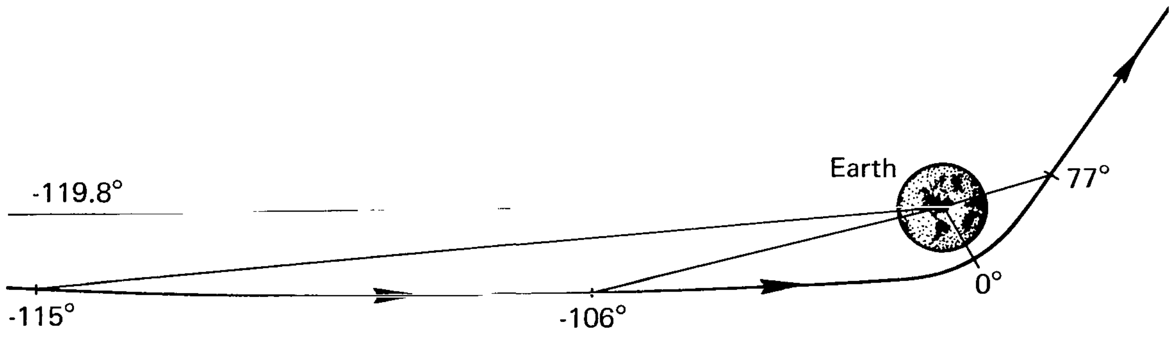


Figure 5.- True anomalies of correction points on hyperbola.

Figures 6(a), 6(b), and 6(c) present the rms errors resulting from velocity corrections made while the spacecraft is approaching perigee on the hyperbola. The results, which are quite similar to those for the return portion of the high (0.98) eccentricity ellipse, indicate that the linear method is superior over most of the trajectory.

In figure 6(a), the error for the linear method is below the scale of the graph, but as the correction point moves nearer to perigee (figs. 6(b) and 6(c)), this error becomes progressively larger. On the other hand, the error for the half-linear method decreases for corrections near perigee. Also, as the correction point moves nearer perigee, the curves for the different values of σ_V become closer together, and in figure 6(c) the curve for $\sigma_V = 0.1$ m/sec is so close to that for $\sigma_V = 0$ that it has been omitted.

As in the case of the ellipses, the error for the linear method becomes larger compared to those for the half-linear method as $\Delta\theta$ decreases. This trend can be seen clearly in figure 6(c) and was also noted in data (not shown) for the other two cases.

Figures 6(d) and 6(e) illustrate the data for the outbound portion of the hyperbola. In both figures, the curve of half-linear error for $\sigma_V = 0.1$ m/sec has been omitted because it is so close to that for $\sigma_V = 0$.

Figure 6(d) shows the data for $\theta_o = 0$. In this case, the half-linear method is superior for all values of θ_f, although the ratio of linear-to-half-linear errors decreases for larger values of $\Delta\theta$. However, the errors in this case will be relatively small for initial deviations likely to be encountered.

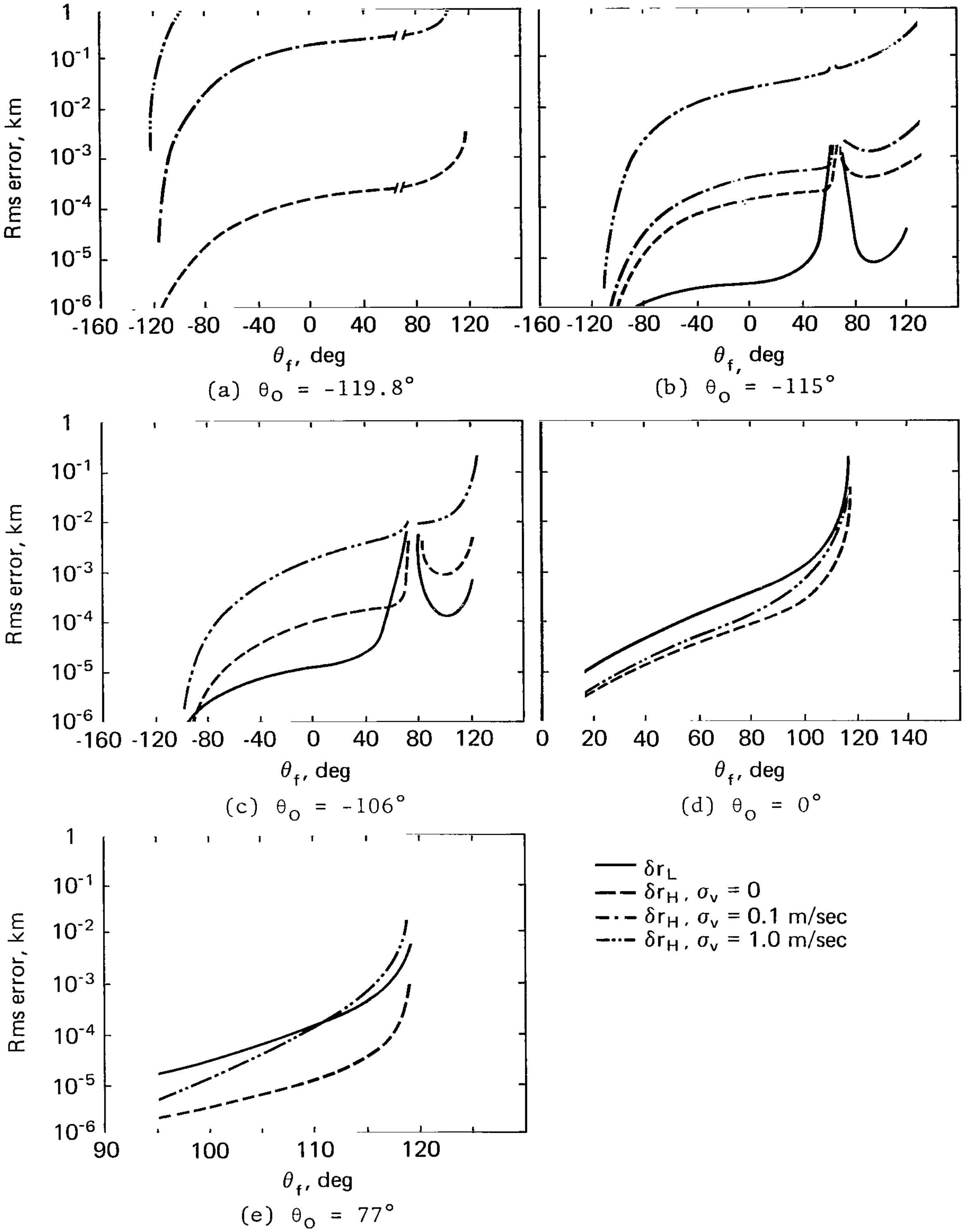


Figure 6.- Rms second-order errors for hyperbola with σ_r = 1 km.

Figure 6(e) shows the result of increasing θ_o to 77°, corresponding to a radial distance from the Earth of about twice the perigee radius. For $\sigma_V = 0$ the half-linear method is superior for all θ_f, but for $\sigma_V = 1.0$ m/sec the half-linear error exceeds that for the linear method when $\theta_f > 111°$. Note that the second-order errors are smaller than in figure 6(d) since the portion of the trajectory between θ_o and θ_f is more nearly a straight line.

The data for the hyperbola show the linear method to be greatly superior on the inbound portion of the hyperbola, while the half-linear method is better, in most cases, on the outbound leg. However, the second-order errors on the outbound leg are comparatively small for the initial errors likely to be encountered, and the linear method is sufficiently accurate. Therefore, it seems reasonable to use the linear method for both parts of the hyperbola.

The rms errors for the conics just presented show that:

1. The linear method is usually the superior one, and the half-linear error is often several orders of magnitude greater than that for the linear method.

2. When the errors for the linear method are the larger, they never exceed the half-linear errors by more than a factor of 5. This situation occurs only where the second-order errors are small or near the singular points, where a velocity correction is unlikely.

3. The half-linear method becomes poorer compared to the linear method as $\Delta\theta$ and σ_V are increased and as the correction point moves away from perigee.

4. The portion of the half-linear error due to initial velocity deviations ranges from a small fraction of that due to position deviations alone to several orders of magnitude greater. However, in practical cases the contribution of the initial velocity deviation will usually be dominant.

Accuracy of Equations for rms Second-Order Errors

The accuracy of equations (6) used in predicting the rms second-order errors for the linear and half-linear methods was checked at several points on the conic trajectory using a Monte Carlo method. The initial deviations for each Monte Carlo sample were obtained by multiplying random numbers from a Gaussian distribution with zero mean and unit standard deviation by σ_r or σ_V, whichever was appropriate. The velocity correction appropriate for the resulting set of initial deviations was computed by both the linear and half-linear methods. Then each of the corrected trajectories was projected (using nonlinear equations) to the terminal point and the terminal error was found.

Since equations (6) are valid only for second-order errors, it was necessary to insure that the Monte Carlo data would not contain significant errors of higher order. This was done as follows: For each pair of correction correction and terminal points, the values of σ_r and σ_V were adjusted by

trial and error to obtain values for which errors of higher than second order could be neglected. The rms error for a small number of samples was computed several times using the same random numbers with different values of σ_r and σ_V. With the resulting data, it was possible to select values of σ_r and σ_V within a range of values where the rms second-order errors increased linearly with σ_r^2 and σ_V^2, and where there was no serious loss in significant figures.

Once good values were established for σ_r and σ_V, the mean-square and rms second-order errors were computed for 100 Monte Carlo samples. The largest difference between the rms error predicted using equations (6) and (7) and that obtained by the Monte Carlo method was about 16 percent of the predicted value. The mean-square error for the Monte Carlo method differed from the predicted value (eqs. (6)) by no more than 33 percent of the predicted value. This appears to be a reasonable agreement for the number of samples used if we consider the degenerate case in which two of the components of both $\overline{r}_0$ and $\overline{V}_0$ are zero. In this case, δr_L and δr_H are members of the chi-squared distribution of random variables, and the standard deviation of the sample mean-square value for 100 samples is 32.7 percent of the population mean-square value. However, even errors of 16 percent in predicting the rms second-order errors are not particularly significant in comparing the linear and half-linear methods since differences of several orders of magnitude may exist.

COMPARISON OF TWO-BODY AND FOUR-BODY RESULTS

Since actual space trajectories are never truly conics we would like to consider the problem of predicting the effects of second-order errors for n-body trajectories involving gravitational anomalies such as oblateness. Equations (6) and (7) could be solved to provide this information, but much more computer time would be required than in the two-body case. Since only qualitative information is needed to determine whether the linear or half-linear method is better, we ask whether the two-body data presented in the previous section can be used for this purpose.

The patched conic approach, which is often used to obtain qualitative information about n-body trajectories, was tried for the qualitative evaluation of second-order errors. A four-body simulated lunar mission was used as a test problem since translunar and transearth trajectories are generally recognized as stringent tests of two-body approximations.

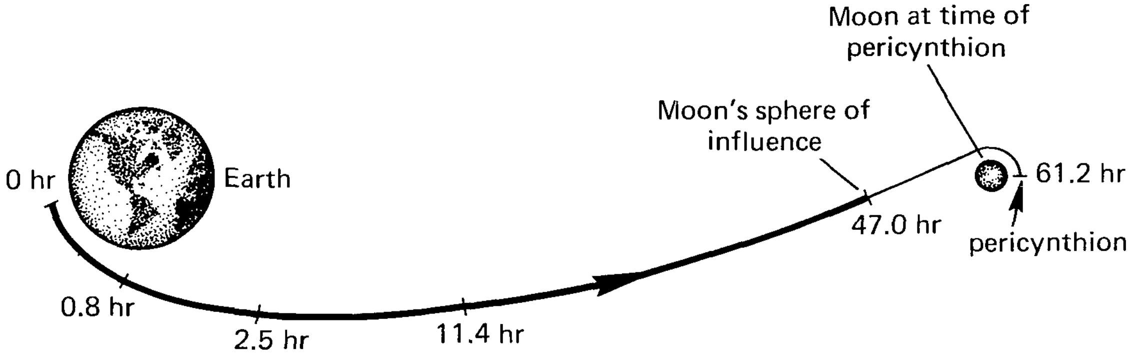
When the lunar trajectory lies entirely inside or outside the Moon's sphere of influence, it can be approximated fairly accurately by a conic. In this case, a direct numerical comparison was made between the Monte Carlo rms errors from the four-body trajectory and the second-order errors from the two-body study. Two conics are required to approximate trajectories crossing the Moon's sphere of influence, and it is not apparent how the second-order errors for the two conics can be combined to give meaningful numerical answers. However, data will be presented under "Discussion of Data" which show that the

trends in the four-body errors are what would be expected from considering data for the two-body cases and support the use of two-body data as presented in the previous section.

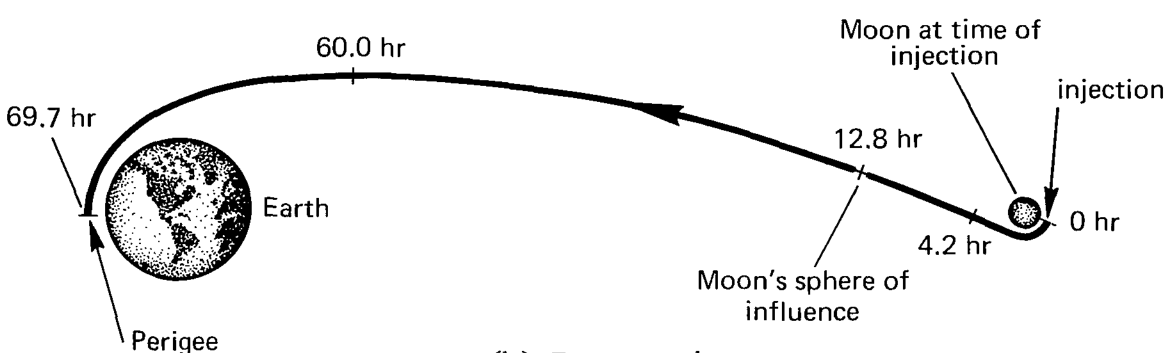
The three conics presented in the two-body section are representative of trajectories encountered in a large number of space missions. Therefore, two-body data can be used to indicate whether the linear or half-linear method is better on most missions.

Description of Simulated Lunar Mission

One translunar and one transearth trajectory that include gravitational forces of the Earth through the second harmonic term, the Moon, and the Sun were considered (fig. 7). Translunar injection occurs at a true anomaly of



(a) Translunar.



(b) Transearth.

Figure 7.- Lunar trajectories in inertial coordinates (not to scale).

about 13° and flight time to pericynthion is 61.2 hours. The Moon's sphere of influence is entered at 47 hours and a true anomaly of about 168°. Transearth injection is at a true anomaly of about 11° and the flight time to return perigee is 69.7 hours with exit from the Moon's sphere of influence at 12.8 hours after injection. These times along with other times used for velocity corrections are indicated in the figure. The heavy black lines indicate the portions of the trajectory over which direct numerical comparison of two-body and four-body results is made.

Three velocity correction times were used for the translunar trajectory: injection time, and 0.8 and 11.4 hours after injection. The latter two times were chosen to represent the earliest and latest times likely for the first midcourse velocity correction. The correction times used for the transearth case were injection, 12.8 hours after injection and, to represent the first midcourse velocity correction, 4.2 hours after injection. Terminal points at 0.8 and 2.5 hours on the translunar trajectory and 60 hours on the transearth trajectory were included to show the effect of increasing $\Delta\theta$.

Computation of Data

Four-body data.- The transition matrices from Danby's method were corrected for perturbing forces by the method of reference 5, and the rms second-order errors for the four-body trajectories were computed by the Monte Carlo method described in the section on two-body results. The Monte Carlo procedure adjusted the magnitudes of σ_r and σ_V so that errors of higher than second order and those due to the loss of significant figures were negligible. The resulting data were normalized to values corresponding to $\sigma_r = 1$ km and $\sigma_V = 0$ or $\sigma_V = 1$ m/sec and tabulated in tables I and II.

Two-body data.- In the cases where the portion of the n-body trajectory between the velocity correction and terminal points lies entirely inside or outside the Moon's sphere of influence, the errors computed for the two-body case were tabulated for comparison. These data were obtained from those presented in the section on two-body results by interpolation. The errors for a given value of θ_f were read from the curves (figs. 3 and 6) and plotted as a function of θ_o. The error data presented in this section were taken from the resulting plot.

Since the mass of the Earth was used in the two-body study, the hyperbola does not represent a Moon-centered trajectory. The data were scaled to fit the osculating hyperbola at transearth injection by changing the units of length from 1 to 0.3 km and the units of time from 1.0 to 1.5 second. A crude correction was also made to account for the fact that the eccentricity of the osculating hyperbola at transearth injection was about 1.6 instead of 2.0 as in the two-body study. To accomplish this, each true anomaly from the osculating hyperbola was multiplied by the ratio of the maximum true anomaly for an eccentricity of 2.0 to that for an eccentricity of 1.6. The same interpolation procedure used for the two-body data from the ellipse was applied to that from the hyperbola.

Discussion of Data

Table I compares the terminal errors for the four-body translunar trajectory with those predicted from the two-body data. The two-body and four-body errors differ at most by a factor of 5, and, since only six Monte Carlo samples were obtained, this is reasonably good agreement.

The linear method is superior for the velocity corrections made at points likely for the first midcourse correction with the terminal point at the Moon's sphere of influence. These are the same results as obtained in the two-body study in which the linear method was superior for the inbound leg of the hyperbola. Therefore, we would expect the linear method to be superior when the ellipse and hyperbola are combined to represent the translunar trajectory; this reasoning is verified by the four-body data.

We noted in examining the two-body data that as $\Delta\theta$ increases the half-linear errors become larger compared to those for the linear method, and the errors for both methods increase in magnitude. The four-body data show that both of these trends continue when the translunar trajectory crosses the Moon's sphere of influence.

Table II presents a similar set of data for the transearth trajectory using the same sets of initial deviations as were used to obtain the data in table I. Again, where direct comparison is possible, there is reasonably good agreement between the two-body and four-body data. On the hyperbolic (within the Moon's sphere of influence) portion of the trajectory, the linear method is superior for $\sigma_V = 1$, but for $\sigma_V = 0$ the half-linear method is best.

The same continuation of trends across the boundary of the Moon's sphere of influence as in the translunar case is seen here. That is, as $\Delta\theta$ increases, the ratio of half-linear-to-linear errors increases, and the magnitudes of the second-order errors increase for both methods. On the hyperbolic part of the trajectory, the linear method is much poorer compared to the half-linear than in the translunar case. However, the elliptic portion constitutes most of the trajectory so that the linear method is still far superior for the complete transearth trajectory.

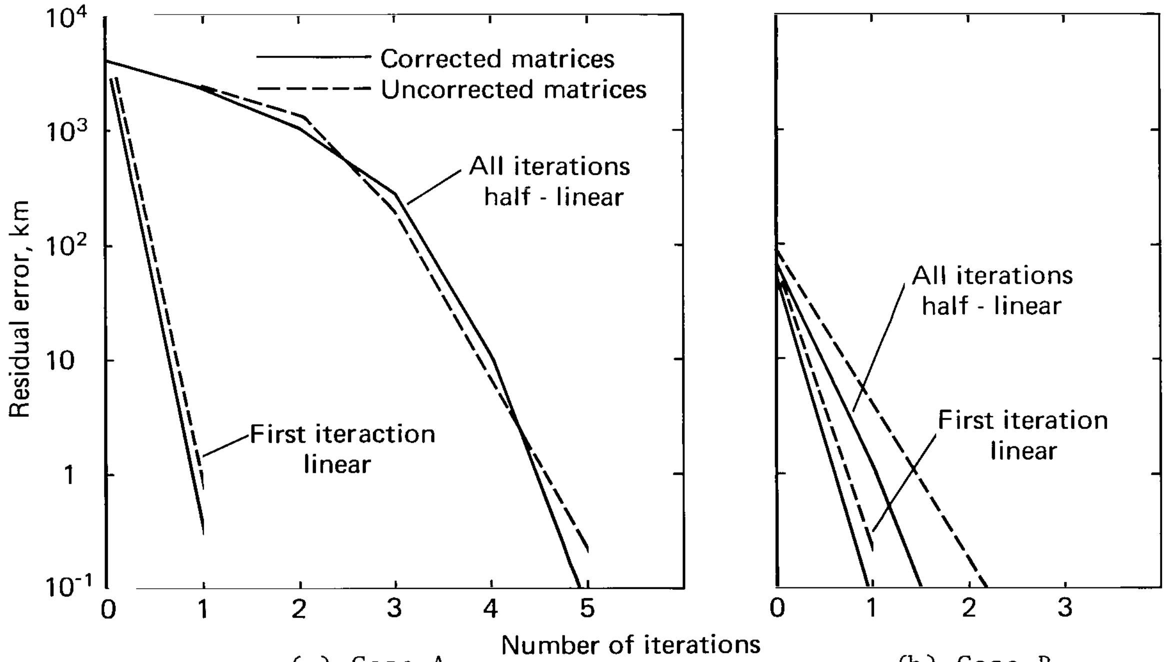
ITERATION CONVERGENCE RATES

If, after the velocity correction is applied, the residual error (δr_L or δr_H) is unacceptable, we may wish to reduce it by iteration. Since the velocity correction in the linear method depends only on ϕ and $\overline{r}_0$, which are unchanged after the first correction, this method cannot be used iteratively. Further, corrections must be computed using the half-linear method, but the efficiency of the first correction greatly influences the rate with which the subsequent iterations converge.

The next data to be presented show how the rates of convergence of iterations are affected by: (a) the method (linear or half-linear) of computing the first trial velocity correction, (b) the magnitude of the initial errors, and (c) the effects of perturbing forces. These data were obtained using two sets of initial deviations selected from the six sets used for the Monte Carlo data in the previous section. Case A yielded the maximum terminal error and case B, the minimum value, when the Monte Carlo trajectories were integrated from injection to pericynthion without a velocity correction. The same sets of initial errors also gave maximum and minimum terminal errors for the transearth trajectory.

The trajectories were started at transearth or translunar injection with these (case A or B) initial errors and integrated ahead to the appropriate velocity correction time. Iterative velocity corrections were continued until the terminal error at perigee or pericynthion was reduced below 1 km. Data were obtained in each case using the transition matrices that had been corrected for perturbing forces and also using the uncorrected matrices. The residual errors after each iteration are plotted in figures 8 through 11 and the residual error corresponding to an abscissa of zero is the terminal error that results if no velocity correction is made.

Figure 8 shows the residual errors in the iteration process for σ_r = 1 km, σ_V = 1 m/sec, and a correction time of 0.8 hour after translunar injection. In both cases, the differences between the residual errors when

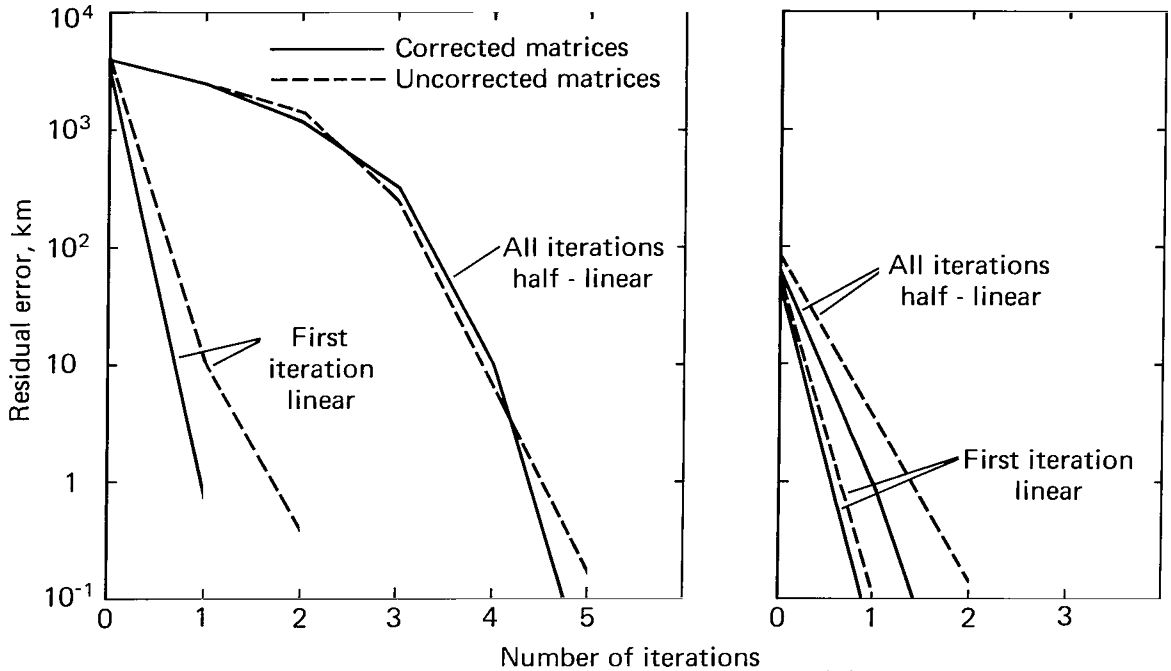


(a) Case A. (b) Case B.

Figure 8.- Errors in iteration for correction 0.8 hours after translunar injection; σ_r = 1 km and σ_V = 1 m/sec.

the corrected transition matrices were used and those for the uncorrected matrices are relatively small except after the last iteration. For both sets of initial errors, the correction computed by the linear method reduces the terminal error below 1 km and no further iteration is necessary. With the first correction by the half-linear method, however, two iterations are required for case B and five iterations for case A.

Figure 9 presents the results for the same initial errors when the velocity correction is made 11.4 hours after injection. The curves in figure 9 are nearly identical to those in figure 8 except for case A with the linear method and uncorrected transition matrices used for the first correction. In this case, one additional iteration was required because of the error due to perturbing forces.

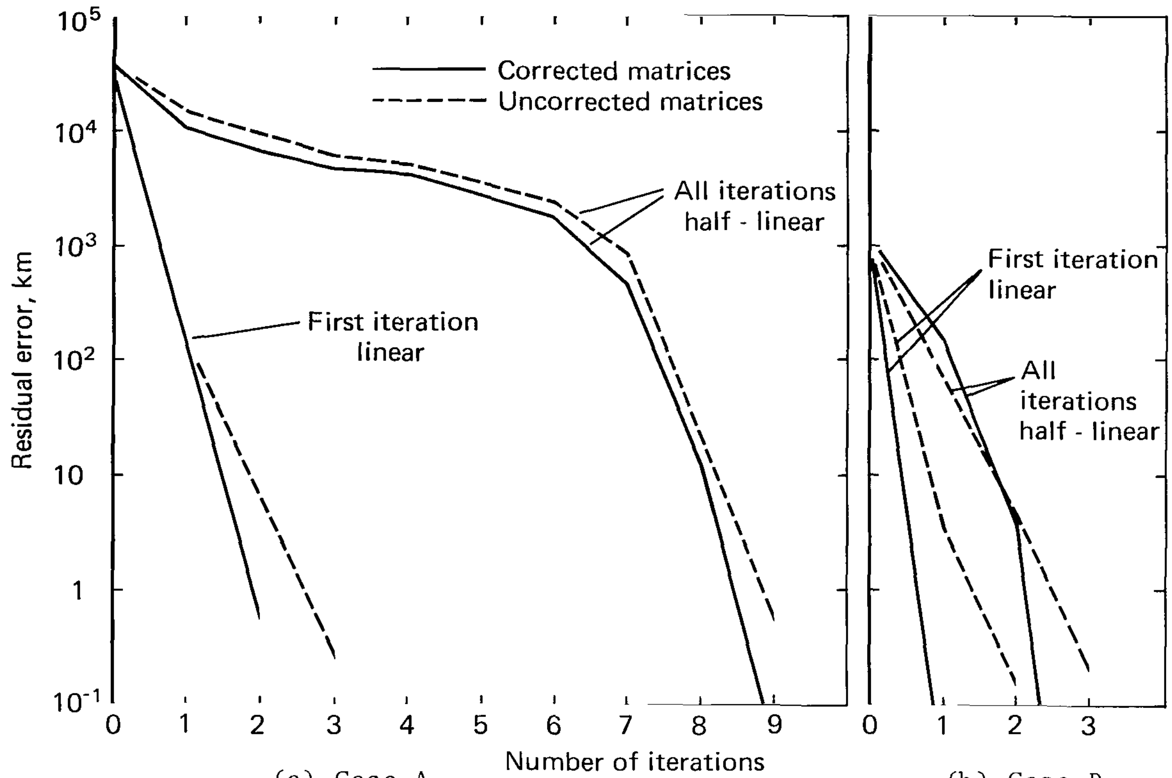


(a) Case A.

(b) Case B.

Figure 9.- Errors in iteration for correction 11.4 hours after translunar injection; σ_r = 1 km and σ_V = 1 m/sec.

Figure 10 shows the same data for the same conditions as figure 9 except that σ_r and σ_V have been increased by an order of magnitude. For case A, using the corrected transition matrices, the total number of iterations increased from one to two when the first correction was linear and from five to nine for a half-linear first correction. For case B, the linear correction still reduced the residual below 1 km, but when the first correction was half-linear the number of iterations rose from two to three. In figure 10, the effects of the perturbing forces are similar to those in figure 9, though somewhat larger, and these errors require an additional iteration in two of the four cases shown. Note that after the first iteration for case A the residual error for the linear method is about 100 times the equivalent error in figure 9, while that for the half-linear method has increased by only a



(a) Case A. (b) Case B.

Figure 10.- Errors in iteration for correction 11.4 hours after translunar injection; σ_r = 10 km and σ_V = 10 m/sec.

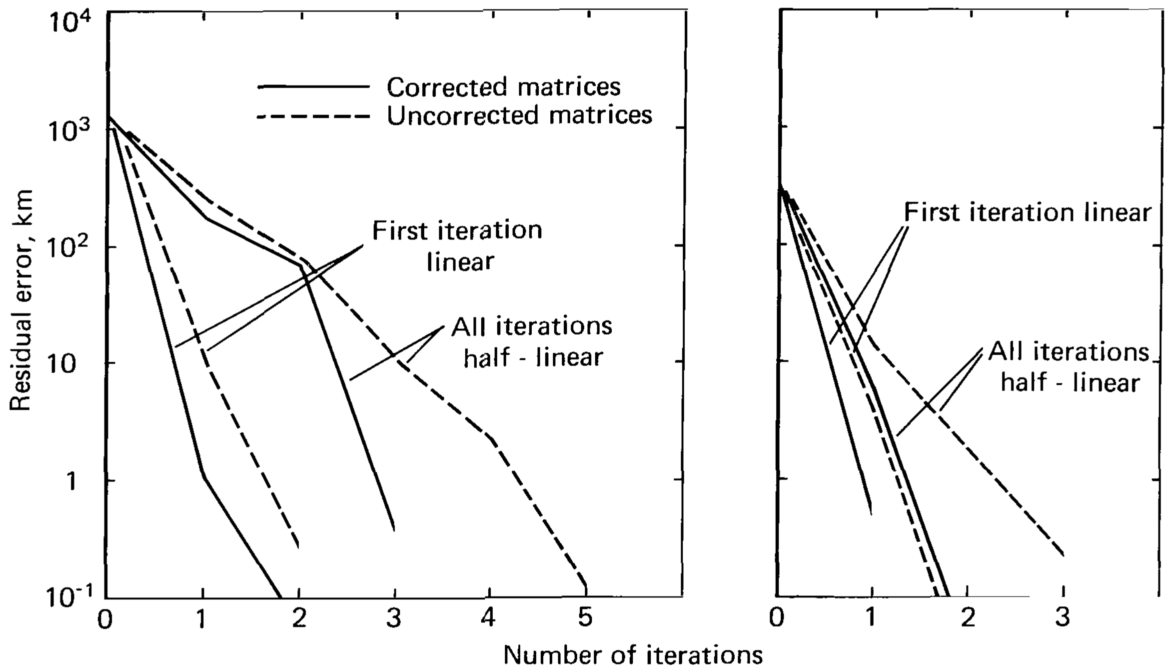
factor of 4. Thus, the assumption that the error due to nonlinearity can be approximated by the second-order term is not valid for the half-linear method and the large case A initial deviations. However, the half-linear error still exceeds the linear error by more than two orders of magnitude, and still illustrates the superiority of the linear method.

Figure 11 presents similar data for the transearth trajectory with σ_r = 1 km, σ_V = 1 m/sec, and the velocity correction 4.2 hours after injection. In this instance, the perturbing forces are somewhat more significant than in the translunar case, but they become dominant only after the terminal error has been reduced to about 10 km. When σ_r and σ_V were increased by an order of magnitude in case A, one additional iteration was required after an initial linear correction; but when the first correction was half-linear, the iteration failed.[1]

The data presented in this section show that iteration is much less likely to be needed if the first velocity correction is made by the linear method. Also, if iteration is necessary, convergence will be faster than when the first correction is made by the half-linear method. As the magnitudes of

[1]If the iteration failed for both methods of making the initial correction, it would be necessary to use nonlinear methods such as those described in references 2 and 6.

the initial deviations increase, the number of iterations required increases for both methods, but the increase is greater when the first correction is half-linear. The errors due to neglecting perturbing forces in computing the transition matrices become important only when the residual errors are small, and they usually cause no more than one additional iteration.



(a) Case A. (b) Case B.

Figure 11.- Errors in iteration for correction 4.2 hours after transearth injection; σ_r = 1 km and σ_V = 1 m/sec.

CONCLUDING REMARKS

The two-body data show that the linear method of velocity correction is usually superior, the second-order errors often being several orders of magnitude less than those for the half-linear method. When the error for the linear method is the larger, it never exceeds the half-linear error by more than a factor of 5. This situation occurs near the singular points, where a correction is unlikely, and for small values of $\Delta\theta$ or on the outbound leg of a hyperbola, where the second-order errors for both methods are small.

The linear method is superior even when there are no initial velocity deviations; but if initial velocity deviations are present the half-linear error is much larger. In fact, the part of the half-linear error caused by initial velocity deviations is usually much larger than that due to initial position deviations.

The comparison of two-body and four-body data showed that, when the portion of the trajectory under consideration can be approximated by a conic, the two-body error analysis is fairly accurate. If a single conic is not

sufficient to approximate the trajectory, the four-body data still follow the same trends as the two-body data, and the linear method is usually superior.

When the linear method is used to compute the first velocity correction, it is much less likely that subsequent iterations will be needed than when the first correction is half linear. If further iteration is necessary, convergence is faster when the first correction is linear. Also, the number of iterations does not increase as rapidly with the magnitude of the initial deviations as with a half-linear correction.

The errors caused by neglecting the perturbing forces in the computation of the transition matrices can be important enough to require an additional iteration or two. However, these errors are much less troublesome than the second-order errors.

We have not arrived at a satisfactory physical or mathematical explanation for the general superiority of the linear method. However, numerical data have been presented for conics representative of those found in most space missions as well as data which show that conics can be used to indicate trends for n-body trajectories. These data are sufficient to show that for most practical applications the linear method should be used.

Ames Research Center
National Aeronautics and Space Administration
Moffett Field, Calif., 94035, Nov. 6, 1968
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APPENDIX A

DERIVATION OF EXPRESSIONS FOR SECOND-ORDER ERRORS

Expressions for the mean-square second-order errors will be derived without the restrictions on the guidance law and statistical distribution of the initial errors. Then, restrictions will be applied to obtain the expressions in the text.

Let $\overline{P}$ be a vector of parameters which are to be constrained to predetermined (reference) values by the guidance system. If $\overline{P}$ is expanded in a Taylor's series about the reference value, then its deviation, $\overline{p}$, from the reference value is

$$\overline{p} = A\overline{r}_o + B\overline{v}_o + \delta\overline{p} \tag{A1}$$

where A and B are matrices of the first partial derivatives of $\overline{P}$ with respect to the Cartesian components of initial position and velocity, and $\delta\overline{p}$ is the summation of the higher order terms in the expansion.

We wish to make a velocity correction, or, in other words, to replace $\overline{v}_o$ with a corrected velocity deviation, $\overline{v}_1$, which will make $\overline{p}$ zero. In computing the velocity correction by the linear method, we assume $\delta\overline{p}$ to be zero. It is shown in reference 4 that a solution for the linear part of equation (A1) is possible if the number of components of $\overline{P}$ does not exceed the number of components in $\overline{v}_o$. Then, with the minimum velocity correction,

$$\overline{v}_1 = -(B^+A)\overline{r}_o + (I-B^+B)\overline{v}_o \tag{A2}$$

where B^+ is the pseudoinverse of B.

When $\overline{v}_o$ is replaced by $\overline{v}_1$, we find that the error in $\overline{P}$ after a correction by the linear method is $\delta\overline{p}_L$, residual of the expansion of $\overline{P}$ for the corrected trajectory about the reference value. Note that $\overline{v}_o$ is replaced with $\overline{v}_1$ so that $\delta_{\overline{p}_L}$ is not the same as $\delta_{\overline{p}}$ in equation (A1).

If we linearize around the actual trajectory instead of the reference, we use the nonlinear equations relating $\overline{P}$ to the initial state and determine $\overline{p}$ by differencing the actual and reference values of $\overline{P}$. Since we can only change the initial velocity (not the position), we wish to compute a change, $\overline{v}_D$, in the initial velocity that will reduce $\overline{p}$ to zero. If $\overline{v}_D$ were known, we could write

$$-\overline{p} = (B+\Delta B)\overline{v}_D + \delta\overline{p}_D \tag{A3}$$

where $\delta\overline{p}_D$ represents the nonlinear terms in the expansion of the desired value of $\overline{P}$ about its actual (uncorrected) value. The term ΔB is added to B to emphasize the fact that the Taylor's series expansion is about the actual trajectory instead of the reference. In computing the velocity correction, we again assume that $\overline{p}$ is due entirely to the first-order term. The same derivation as given for the linear method in reference 4 gives the correction that will reduce the linear portion of $\overline{p}$ to zero as

$$\overline{v}_H = -(B+\Delta B)^+\overline{p} = -(B^++C)\overline{p} \tag{A4}$$

where

$$C = (B+\Delta B)^+ - B^+ \tag{A5}$$

Expansion of the trajectory corrected by the half-linear method about the actual trajectory gives the deviation of $\overline{P}$ from its uncorrected value as

$$\overline{p}_H = -\overline{p} + \delta\overline{p}_H \tag{A6}$$

where $\delta\overline{p}_H$ represents the nonlinear terms in the expansion of the corrected $\overline{P}$ about the actual one. Since we wish this deviation to be exactly $-\overline{p}$, the residual error after the half-linear correction is $\delta\overline{p}_H$.

We will assume that $\delta\overline{p}_L$ and $\delta\overline{p}_H$ can be approximated accurately by the second-order terms in the appropriate Taylor's series, and in the remainder of the derivation they will be used to denote only the second-order terms. The second-order term in the expansion of the ith component, α, of $\overline{P}$ about its reference value is given by

$$\delta\alpha = \frac{1}{2}\begin{pmatrix}\overline{r}_o^T & \overline{v}_o^T\end{pmatrix} G \begin{pmatrix}\overline{r}_o \\ \overline{v}_o\end{pmatrix} \tag{A7}$$

where G is the 6 by 6 matrix of the second partial derivatives of α with respect to the components of the initial position and velocity.

If we wish to compute the second-order error after a linear correction, we substitute $\overline{v}_1$ for $\overline{v}_o$ in equation (A7). Since we can write $\overline{v}_1$ as a linear function of $\overline{r}_o$ and $\overline{v}_o$, we obtain

$$\delta\alpha_L = \frac{1}{2}\begin{pmatrix}\overline{r}_o^T & \overline{v}_o^T\end{pmatrix}\begin{bmatrix} I & (-B^+A)^T \\ 0 & (I-B^+B)\end{bmatrix} G \begin{bmatrix} I & 0 \\ (-B^+A) & (I-B^+B)\end{bmatrix}\begin{pmatrix}\overline{r}_o \\ \overline{v}_o\end{pmatrix} \tag{A8}$$

For the half-linear system, we can write

$$\delta\alpha_H = \frac{1}{2}\begin{pmatrix} 0 & \bar{v}_H{}^T \end{pmatrix} (G+\Delta G) \begin{pmatrix} 0 \\ \\ \bar{v}_H \end{pmatrix} \tag{A9}$$

where the term ΔG emphasizes the fact that we are expanding about the actual trajectory instead of the reference. Since $\bar{v}_H{}^T \Delta G \bar{v}_H$ is of third order, it will be neglected in the computation of the second-order error. Substituting for $\bar{p}$ in equation (A4) from (A1) gives

$$\bar{v}_H = -(B^+ + C)(A\bar{r}_o + B\bar{v}_o + \delta\bar{p}) \tag{A10}$$

Since $C\bar{p}$ and $B^+\delta p$ are themselves of second order they will be neglected in the computation of $\delta\alpha_H$ so that

$$\bar{v}_H \approx -B^+ A\bar{r}_o - B^+ B\bar{v}_o \tag{A11}$$

These simplifications give the half-linear error as a function of $\bar{r}_o$ and $\bar{v}_o$ as follows

$$\delta\alpha_H = \frac{1}{2}\begin{pmatrix} \bar{r}_o^T & \bar{v}_o^T \end{pmatrix} \begin{bmatrix} 0 & (-B^+A)^T \\ \\ 0 & (-B^+B) \end{bmatrix} G \begin{bmatrix} 0 & 0 \\ \\ -(B^+A) & (-B^+B) \end{bmatrix} \begin{pmatrix} \bar{r}_o \\ \\ \bar{v}_o \end{pmatrix} \tag{A12}$$

Note that equations (A8) and (A12) are both of the form

$$\delta\alpha = \frac{1}{2} Hx \tag{A13}$$

where

$$x = \begin{pmatrix} \bar{r}_o \\ \bar{v}_o \end{pmatrix}$$

Therefore,

$$\delta\alpha = \frac{1}{2}\sum_{i=1}^{6}\sum_{j=1}^{6} h_{ij} x_i x_j$$

so that

$$\left.\begin{aligned} E(\delta\alpha^2) &= \frac{1}{4} E\left[\sum_{i=1}^{6} \sum_{j=1}^{6} \sum_{k=1}^{6} \sum_{l=1}^{6} h_{ij}h_{kl}(x_i x_j x_k x_l)\right] \\ &= \frac{1}{4} \sum_{i=1}^{6} \sum_{j=1}^{6} \sum_{k=1}^{6} \sum_{l=1}^{6} h_{ij}h_{kl}E(x_i x_j x_k x_l) \end{aligned}\right\} \quad \text{(A14)}$$

where E indicates the expected value. Equation (A14) can usually be simplified when the statistical distributions of the components of x are known.

At this point we will assume that the components of x are independent and Gaussian with zero mean and standard deviation σ_i. For this distribution,

$$Ex_i^n x_j^m = (Ex_i^n)(Ex_j^m)$$

and, if n is odd,

$$Ex_i^n = 0$$

Therefore, we need consider only those terms in equation (A14) that contain only even powers of the components of x. In this case,

$$E(\delta\alpha^2) = \frac{1}{4} E\left(\sum_{i=1}^{6} \sum_{j=1}^{6} h_{ii}h_{jj}x_i^2 x_j^2\right) + \frac{1}{2} E\left(\sum_{i=1}^{6} \sum_{j=1}^{6} h_{ij}^2 x_i^2 x_j^2\right)_{i\neq j}$$

(note that H is symmetric). Since

$$Ex_i^2 x_j^2 = \sigma_i^2 \sigma_j^2$$

and

$$Ex_i^4 = 3\sigma_i^4$$

$$E(\delta\alpha^2) = \frac{1}{4}\left(\sum_{i=1}^{6} \sum_{j=1}^{6} h_{ii}h_{jj}\sigma_i^2\sigma_j^2\right)_{i\neq j} + \frac{3}{4} \sum_{i=1}^{6} h_{ii}^2 \sigma_i^4 + \frac{1}{2}\left(\sum_{i=1}^{6} \sum_{j=1}^{6} h_{ij}^2\sigma_i^2\sigma_j^2\right)_{i\neq j}$$

or

$$E(\delta\alpha^2) = \frac{1}{2}\sum_{i=1}^{6}\sum_{j=1}^{6} h_{ij}^2\sigma_i^2\sigma_j^2 + \frac{1}{4}\sum_{i=1}^{6}\sum_{j=1}^{6} h_{ii}h_{jj}\sigma_i^2\sigma_j^2$$

This equation can be reduced to

$$E(\delta\alpha^2) = \frac{1}{2}\,\mathrm{Tr}(HDHD) + \frac{1}{4}\,(\mathrm{Tr}HD)^2 \tag{A15}$$

where D is the diagonal matrix whose diagonal terms are σ_i^2. If we make the further simplification that the standard deviations for the components of position and velocity, respectively, are equal, then

$$HD = \begin{pmatrix} H_1 & H_2 \\ H_3 & H_4 \end{pmatrix} \begin{bmatrix} \sigma_r^2 I & 0 \\ 0 & \sigma_v^2 I \end{bmatrix}$$

$$\left. E(\delta\alpha^2) = \frac{1}{2}\,\sigma_r^4\|H_1\|_E^2 + \sigma_r^2\sigma_v^2\|H_2\|_E^2 + \frac{1}{2}\,\sigma_v^4\|H_4\|^2 + \frac{1}{4}\left[\mathrm{Tr}\left(\sigma_r^2 H_1 + \sigma_v^2 H_4\right)\right]^2 \right\} \tag{A16}$$

The expanded form of the H_i from equations (A4) and (A12) is summarized in tabular form below where

$$G = \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix}$$

Submatrix	Linear	Half-linear
H_1	$G_1 - 2G_2B^+A + (B^+A)^T G_4 B^+A$	$(B^+A)^T G_4 B^+A$
$H_2 = H_3{}^T$	$G_2(I-B^+B) - (B^+A)^T G_4(I-B^+)$	$(B^+A)^T G_4 B^+B$
H_4	$(I-B^+B)G_2(I-B^+B)$	$B^+BG_4B^+B$

If we expand equation (A9) we find that wherever $\overline{v}_o$ occurs in the expression for $\delta\alpha_L$ it is always multiplied by $(I-B^+B)$. On the other hand, in the expression for $\delta\alpha_H$ given by expanding equation (A12), $\overline{v}_o$ is always multiplied by $-B^+B$. Note that $B^+B\overline{v}_o$ represents the components of $\overline{v}_o$ (see ref. 4) that affect $\overline{P}$, while $(I-B^+B)\overline{v}_o$ is orthogonal to the first set and does not affect $\overline{P}$.

Now consider the three possible types of velocity correction discussed in reference 4.

(1) The rank of B is 3. In this case, $B^+B = I$ so that for the linear method $H_2 = H_3 = H_4 = 0$ and α_L is not a function of $\overline{v}_0$, while for the half-linear method all the terms in equation (A16) are present.

(2) The rank of B is 2. In this case, H_2, H_3, and H_4 are of rank 1 for the linear method, while for the half-linear method they are of rank 2. This means that for the linear method the second-order errors are a function of the one component of $\overline{v}_0$ that does not affect the parameters we wish to control. On the other hand, for the half-linear method, the second-order errors are a function of the two components of $\overline{v}_0$ that do affect the parameters.

(3) The rank of B is 1. In this case, two components of $\overline{v}_0$ contribute to the error in the linear method, while only one contributes to the errors in the half-linear method.

The above shows that the influence of the initial veloctiy deviation on the half-linear method decreases with the rank of B, while the reverse is true for the linear method. Thus, we would expect the relative advantage of the linear method to decrease with the number of parameters controlled.

Finally, we wish to restrict the problem to fixed-time-of-arrival guidance. In this case, it is shown in reference 4 that

$$A = \phi_1$$

$$B = \phi_2$$

where

$$\phi = \begin{bmatrix} \phi_1 & \phi_2 \\ \phi_3 & \phi_4 \end{bmatrix}$$

is the state transition matrix relating the terminal errors to $\overline{r}_0$ and $\overline{v}_0$.

For fixed time of arrival using the linear method

$$H_1 = G_1 - 2G_2\phi_2^{-1}\phi_1 + \phi_2^{-1}\phi_1 G_4 \phi_2^{-1}\phi_1$$

$$H_2 = H_3 = H_4 = 0$$

In the text H_1 for the linear method is denoted as H_L.

For the half-linear method,

$$H_1 = \phi_2^{-1}\phi_1 G_4 \phi_2^{-1}\phi_1$$

$$H_2 = H_3^T = \phi_2^{-1}\phi_1 G_4$$

$$H_4 = G_4$$

Substituting these matrices in equation (A16) gives

$$E\left(\delta X_L^2\right) = \frac{1}{2}\,\sigma_r^4 \|H_L\|_E^2 + \frac{1}{4}\,\sigma_r^4\,[\mathrm{Tr} H_L]^2 \tag{A17}$$

and

$$E\left(\delta X_H^2\right) = \frac{1}{2}\left(\sigma_r^4\|H_1\|_E^2 + 2\sigma_r^2\sigma_V^2\|H_2\|_E^2 + \sigma_r^4\|H_4\|_E^2\right) + \frac{1}{4}\left[\mathrm{Tr}\left(\sigma_r^2 H_1 + \sigma_V^2 H_4\right)\right]^2 \tag{A18}$$

Note that $\delta\alpha$ has been replaced by δX since fixed-time-of-arrival controls the Cartesian components of the terminal position vectors. The equations for the Y and Z components are the same as equations (A17) and (A18) except that different matrices G_i of second partial derivatives are used in defining the H_i.

APPENDIX B

COMPUTATION TECHNIQUES

This appendix describes some of the computation techniques used in obtaining the numerical results presented in the report. The discussion is divided into two sections, the first of which deals with the four-body computations used in the comparison of two-body and four-body results and for the study of iteration convergence rates. The second section discusses the computations used in obtaining the two-body results.

Four-Body Computations

The four-body trajectories were integrated using Danby's method and the digital computer program described in reference 1. The version of this program used computes the osculating conic covering each integration step in double precision. The two-body transition matrix over each integration step and the perturbation from the osculating conic are computed in single precision, and rectification occurs after each integration step. The transition matrix over longer time arcs is obtained by multiplying together transition matrices across the individual integration steps. The matrix multiplication subroutine used for this multiplication accumulates each individual element of the product matrix in double precision and then stores it as a single precision number.

The accuracy of the resulting matrix was checked for several example cases, including translunar injection to pericynthion and transearth injection to return perigee, by multiplying the matrix by its own inverse.[1] The resulting product in each case was a unit matrix to within eight significant figures. That is, if we let

$$\phi^{-1}\phi = C$$

then

$$\left. c_{ij} = \sum_{k=1}^{6} \phi_{ik}\phi_{kj}^{-1} \right\} \tag{B1}$$

---

[1]There is no loss of numerical accuracy in computing ϕ^{-1} since

$$\phi^{-1} = \begin{bmatrix} \phi_4^T & -\phi_2^T \\ -\phi_3^T & \phi_1^T \end{bmatrix}$$

and

$$c_{ij} = 1 \pm \varepsilon_{ij}, \qquad i = j$$

$$c_{ij} = \pm\varepsilon_{ij}, \qquad i \neq j$$

where ε_{ij} is less than 10^{-8} times the largest product in the summation in equation (B1). This is the same accuracy obtained by integrating the variational equations of motion using Cowell's method as described in reference 3. Using a matrix multiplication routine without the double precision accumulation feature results in the loss of roughly one significant figure.

For most of the four-body data, the transition matrix over each integration step was corrected to account for the perturbing forces by the method described in reference 6. The time required for this correction was not determined, but 1300 additional words of computer storage were needed. Since this is nearly half the storage required for the entire trajectory integration program without the matrix correction, the correction procedure would probably not be used in practical applications.

When velocity corrections were made by the linear method, the values of $\overline{r}_0$, $\overline{v}_0$, and ϕ were obtained by integrating the reference trajectory backward from the terminal time to the correction time. This approach requires less storage than the method used in reference 3. There, the transition matrix from injection to the terminal time is precomputed, stored, and then updated by post-multiplying by the inverse of the transition matrix from injection to the correction time.

Two-Body Computations

The two-body computations were carried out using the appropriate subroutines from the Danby integration program that have been modified so that all operations including computation of the transition matrices are in double precision. The transition matrices were computed along the reference conic and along a set of conics with initial position and velocity perturbed from the reference. The elements of the transition matrices were differenced to obtain the second partial derivatives. The desired accuracy was achieved by judicious choice of the initial deviations. This method was used for simplicity in programming (the necessary subroutines were already available), and it is not particularly recommended.

An alternate approach that would yield a more efficient program would be to use the closed-form equations for the second partial derivatives given in reference 7 or 8.

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TABLE I.- RMS ERRORS AND RATIOS FOR TRANSLUNAR TRAJECTORIES

Time from injection, hr		Type of data	Second-order errors				
			Linear	Half-linear, $\sigma_V = 0$		Half-linear, $\sigma_V = 1$	
Correction	Terminal		rms value, km	rms value, km	Ratio to linear	rms value, km	Ratio to linear
0.0	0.8	Two-body	1.5×10^{-3}	9.5×10^{-4}	6.3×10^{-1}	9.3×10^{-4}	6.2×10^{-1}
		Four-body	1.1×10^{-3}	6.5×10^{-4}	5.9×10^{-1}	1.1×10^{-3}	1.0
.0	47.0	Two-body	1.4	2.4	1.7	3.5	2.5
		Four-body	6.4×10^{-1}	2.3	3.6	4.6	7.2
.0	61.2	Four-body	1.5	3.1×10^{2}	2.1×10^{2}	1.2×10^{3}	8.0×10^{2}
.8	2.5	Two-body	8.0×10^{-5}	1.3×10^{-5}	1.6×10^{-1}	3.7×10^{-4}	4.6
		Four-body	5.0×10^{-5}	2.7×10^{-6}	5.4×10^{-2}	2.9×10^{-4}	5.8
.8	47.0	Two-body	7.8×10^{-3}	1.2×10^{-2}	1.5	4.2×10^{-1}	5.4×10^{1}
		Four-body	7.1×10^{-3}	2.4×10^{-3}	3.4×10^{-1}	7.0×10^{-1}	9.9×10^{1}
.8	61.2	Four-body	1.0×10^{-2}	6.2×10^{-1}	6.2×10^{1}	1.6×10^{1}	1.6×10^{4}
11.4	47.0	Two-body	1.2×10^{-5}	2.7×10^{-6}	2.3×10^{-1}	1.6×10^{-2}	1.3×10^{3}
		Four-body	1.3×10^{-5}	4.1×10^{-6}	3.2×10^{-1}	2.1×10^{-2}	1.6×10^{3}
11.4	61.2	Four-body	2.2×10^{-5}	2.1×10^{-3}	9.5×10^{1}	3.6×10^{1}	1.6×10^{6}

TABLE II.- RMS ERRORS AND RATIOS FOR TRANSEARTH TRAJECTORIES

Time from injection, hr		Type of data	Second-order errors				
			Linear	Half-linear, $\sigma_V = 0$		Half-linear, $\sigma_V = 1$	
Correction	Terminal		rms value, km	rms value, km	Ratio to linear	rms value, km	Ratio to linear
0.0	12.8	Two-body	3.7×10^{-2}	1.3×10^{-2}	3.5×10^{-1}	4.2×10^{-2}	1.1
		Four-body	4.6×10^{-2}	3.2×10^{-2}	7.0×10^{-1}	9.6×10^{-2}	2.1
.0	69.7	Four-body	1.7	1.5×10^{1}	8.8	1.3×10^{2}	7.6×10^{1}
4.2	12.8	Two-body	2.3×10^{-5}	7.7×10^{-7}	3.4×10^{-2}	1.2×10^{-3}	5.2×10^{1}
		Four-body	7.4×10^{-6}	4.1×10^{-7}	5.5×10^{-2}	4.0×10^{-4}	5.4×10^{1}
4.2	60.0	Four-body	1.2×10^{-4}	1.0×10^{-5}	8.3×10^{-2}	1.8×10^{-1}	1.5×10^{3}
4.2	69.7	Four-body	4.5×10^{-4}	6.1×10^{-3}	1.4×10^{1}	3.2×10^{1}	7.1×10^{4}
12.8	69.7	Two-body	1.1×10^{-4}	5.1×10^{-3}	4.6×10^{1}	1.6×10^{2}	1.4×10^{6}
		Four-body	2.9×10^{-5}	8.1×10^{-3}	2.8×10^{2}	6.8×10^{1}	2.3×10^{6}

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